Winter term 2014/15
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Combinatorial Optimization

Exercise Sheet 9

Exercise 9.1: Prove Tutte's perfect b-matching characterization (Theorem 3.29). Let G be an undirected graph, $u : E(G) \to \mathbb{N} \cap \{\infty\}$, and $b : V(G) \to \mathbb{N}$. (G, u) has a perfect b-matching if and only if for any two subsets $X, Y \subset V(G)$ with $X \cap Y = \emptyset$, the number of connected components C in $G - X - Y$ for which $\sum_{v \in V(C)} b(c) + \sum_{e \in E(V(C),Y)} u(e)$ is odd is upper bounded by

$$\sum_{v \in X} b(v) + \sum_{y \in Y} \left(\sum_{e \in \delta(y)} u(e) - b(y) \right) - \sum_{e \in E(X,Y)} u(e).$$

(4 Points)

Exercise 9.2: Let G be a graph, $b : V(G) \to \mathbb{N}$, and $c : E(G) \to \mathbb{R}$ a weight function

1. Show that the uncapacitated maximum-weight b-matching problem in bipartite graphs can be solved in strongly polynomial time. (2 Points)
2. Use Step 1 to show that the uncapacitated maximum-weight b-matching problem can be solved in strongly polynomial time if b is even. (2 Points)
3. Use Step 2 to show that the uncapacitated maximum-weight b-matching problem can be solved in strongly polynomial time. (2 Points)
4. Use Step 3 to show that the capacitated maximum-weight b-matching problem for edge capacities $u : E(G) \to \mathbb{N} \cup \{\infty\}$ can be solved in strongly polynomial time. (2 Points)

Exercise 9.3: Given a graph G, a 2-cover of G is an assignment $y : V(G) \to \{0, 1, 2\}$ such that $y(v) + y(w) \geq 2$ for all $\{v, w\} \in E(G)$. If y is a 2-cover, the set $\{v \in V(G) : y(v) = 0\}$ is a stable set. Conversely, a stable set $A \subseteq V(G)$ determines a 2-cover y by setting $y(v) = 0$ for all $v \in A$, $y(v) = 2$ for all $v \in \Gamma(A)$, and $y(v) = 1$ for the remaining vertices. Prove:

1. The maximum size of a 2-matching in G equals the minimum size of a 2-cover. (2 Points)

2. G has a perfect 2-matching if and only if $|\Gamma(A)| \geq |A|$ for every independent set $A \subseteq V(G)$. (2 Points)

Deadline: Tuesday, December 9, 2014, before the lecture.
Information: Submissions by groups of one or two students are allowed.