Combinatorial Optimization

Exercise Sheet 5

Exercise 5.1: Let $G = (V, E)$ be an undirected graph and Q its fractional perfect matching polytope, which is defined by

$$Q = \{x \in \mathbb{R}^E : x_e \geq 0 \ (e \in E), \sum_{e \in \delta(v)} x_e = 1 \ (v \in V)\}.$$

Prove that a vector $x \in Q$ is a vertex of Q if and only if there exist vertex disjoint odd circuits $C_1, \ldots, C_k$ and a perfect matching M in $G - (V(C_1) \cup \ldots \cup V(C_k))$ such that

$$x_e = \begin{cases} \frac{1}{2} & \text{if } e \in E(C_1) \cup \ldots \cup E(C_k), \\ 1 & \text{if } e \in M, \\ 0 & \text{otherwise.} \end{cases}$$

These vertices are called half-integral. (Theorem 1.64)   (4 Points)

Exercise 5.2: Show how the following problem can be solved in polynomial time: Given a graph G and edge weights $c : E(G) \to \mathbb{R}_{>0}$, find an edge cover $F \subseteq E(G)$ that minimizes $\sum_{e \in F} c(e)$.   (4 Points)

Exercise 5.3: Let G be an undirected graph with edge weights $c : E(G) \to \mathbb{R}_{>0}$, and let $v, w \in V$ be two distinct vertices. Describe a polynomial-time algorithm that computes, among all v-w-paths having an even number of edges, a path of minimum weight, and prove its correctness.   (4 Points)

Exercise 5.4: Let $k \geq 2$ be an integer and G a k-regular and $(k-1)$-edge-connected graph with an even number of vertices. Let $c : E(G) \to \mathbb{R}_+$. Prove that there is a perfect matching M in G with $c(M) \geq \frac{1}{k}c(E(G))$.
Hint: Consider the perfect matching polytope.   (4 Points)

Note the information given on page 2!

Deadline: Tuesday, November 11, 2014, before the lecture.
Information: Submissions by groups of one or two students are allowed.

Information by the student council: "On November 18, beginning at 18 o'clock, the student council will host a plenary assembly for all math students. These topics will be discussed: interim mensa, improvements of examination regulations and local numerus clausus. Further information on these topics are available at the showcase in the auxiliary building as well as on fsmath.uni-bonn.de. Attend numerously!"