Winter term 2014/15
Professor Dr. Stephan Held
Jannik Silvanus

Research Institute for Discrete Mathematics
University of Bonn

Combinatorial Optimization

Exercise Sheet 2

Exercise 2.1 Let $G = (V, E)$ be a bipartite graph with bipartition

$$V = \{a_1, \ldots, a_k\} \dot{\cup} \{b_1, \ldots, b_k\}.$$

For any vector $x = (x_e)_{e \in E}$, we define a matrix $M_G(x) = (m^x_{ij})_{1 \leq i,j \leq k}$ by

$$m^x_{ij} = \begin{cases} x_e & \text{if } e = \{a_i, b_j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Its determinant $\det M_G(x)$ is a polynomial in x. We further define the permanent of a $k \times k$ matrix M as

$$\operatorname{per}(M) := \sum_{\pi \in S_k} \prod_{i=1}^{k} m_{i\pi(i)},$$

where S_k is the set of permutations of $\{1, \ldots, k\}$. Prove:

(i) G has a perfect matching if and only if $\det M_G(x)$ is not identically 0. (2 Points)

(ii) If G is simple, it has exactly $\operatorname{per}(M_G(1, \ldots, 1))$ perfect matchings. (4 Points)

Exercise 2.2: Let $G = (V, E)$ with $|V| = 2k$ and $|\delta(v)| \geq k$ for all $v \in V$. Show that G has a perfect matching. (2 Points)

Exercise 2.3: Prove that every 3-regular simple graph with at most two bridges has a perfect matching. (3 Points)

Exercise 2.4: Prove that a graph G has a perfect matching if and only if for each $X \subseteq V(G)$, the graph $G - X$ has at most $|X|$ factor-critical components. (3 Points)

Exercise 2.5: Prove: An undirected graph G is 2-edge-connected if and only if $|E(G)| \geq 2$ and G has an ear-decomposition. (2 Points)

Deadline: Tuesday, October 21, 2014, before the lecture.
Information: submissions by groups of one or two students are allowed.