Winter term 2012/13
Juniorprofessor Dr. Stephan Held
Jan Schneider, Daniel Rotter

Research Institute for Discrete Mathematics
University of Bonn

Combinatorial Optimization

Exercise Sheet 14

Exercise 14.1:

1. Prove: A greedoid is a matroid if and only if it is an interval greedoid without lower bounds. (2 Points)
2. Prove: A greedoid is an antimatroid if and only if it is an interval greedoid without upper bounds. (2 Points)

Exercise 14.2:
Let $X = \{x_1, \ldots, x_m\}$ be a finite set of discrete random variables, each taking values in a finite set Y and having probability distribution p. For $S \subseteq \{1, \ldots, m\}$ we define

$$H(S) = -\sum_{z \in Y^{|S|}} p(x_s = z_s \ \forall s \in S) \cdot \log(p(x_s = z_s \ \forall s \in S))$$

to be the *Shannon-Entropy*. Prove that H is submodular. (4 Points)

Exercise 14.3:
Let $G = (V, E)$ be an undirected graph. For a set $X \subseteq V(G)$ let $f(X)$ denote the number of edges in E incident to X.

1. Prove that f is a submodular function. (2 Points)
2. Prove that it is *NP*-hard to find a set $X \subseteq V$ maximizing $f(X)$. (2 Points)

Let $y : V(G) \to \mathbb{N}$ be a function. We want to find an *orientation* of G (i.e. a directed graph G' such that $V(G) = V(G')$ and the underlying undirected graph is equal to G) such that

$$|\delta^-(v)| \ = \ y(v) \text{ for each } v \in V(G). \tag{1}$$

Continued on page 2.

3. Show that such an orientation exists if and only if

$$y(V) = |E| \quad \text{and} \quad y(X) \leq f(X)\ \forall X \subseteq V(G). \tag{2}$$

Hint: Construct a network (V', E', u) with $V' = E \cup V \cup \{s, t\}$, $E' = \{(s, e) \mid e \in E\} \cup \{(v, t) \mid v \in V\} \cup \{(e, v) \mid v \in e\}$ and a suitable capacity function u such that a flow g in (V', E', u) corresponds to an orientation of $\sum_{e \in \delta^+(s)} g(e)$ edges in $E(G)$. Use the MAX-FLOW-MIN-CUT Theorem.

(2 Points)

4. Give a polynomial time combinatorial algorithm which either finds
   - an orientation as desired or
   - a set $X \subseteq V(G)$ which serves as certificate that such an orientation does not exist.

(1 Point)

5. Consider the following alternative algorithm:
   If there is no orientation as desired, stop. Otherwise, start with $G' = (V(G), \varnothing)$. For each edge $e = \{v, w\} \in E(G)$, set $G' := G' + (v, w)$, $G := G - e$ and $y(w) := y(w) - 1$ if there exists an orientation satisfying (1) for $G - e$ after decreasing $y(w)$ by 1. If this is not the case, set $G' := G' + (w, v)$, $G := G - e$, $y(v) := y(v) + 1$.
   The question, if an orientation satisfying (1) exists can be answered by using Schrijver's algorithm to find a set X minimizing the submodular function $f(X) - y(X)$.

   Compare the running-time of this algorithm to the running-time of your algorithm from exercise 14.3.4. (1 Point)

Deadline: Tuesday, January 29, 2013, before the lecture.