Winter term 2012/13 Research Institute for Discrete Mathematics
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Combinatorial Optimization

Exercise Sheet 12

Exercise 12.1:
Let E be a finite set and $f : 2^E \to \mathbb{R}$. Prove that f is submodular if and only if $f(X \cup \{y, z\}) - f(X \cup \{y\}) \leq f(X \cup \{z\}) - f(X)$ for all $X \subseteq E$ and $y, z \in U$ with $y \neq z$.

(4 Points)

Exercise 12.2:
Let G be a directed graph, $s, t \in V(G)$, $u : E(G) \to \mathbb{R}_+$, and $A := \delta^+(S)$. Prove that $P := \{x \in \mathbb{R}^A_+ \,|\, \text{there is an } s\text{-}t\text{-flow } f \text{ in } (G, u) \text{ with } f(e) = x_e \text{ for all } e \in A\}$ is a polymatroid.

(4 Points)

Exercise 12.3:
Prove that the set of vertices of the base polyhedron of a submodular function $f : 2^E \to \mathbb{R}$ with $f(\varnothing) = 0$ is precisely the set of vectors $b^{\prec}$ for all total orders $\prec$ of E, where $b^{\prec}(e) := f(\{v \in E \,|\, v \preceq e\}) - f(\{v \in E \,|\, v \prec e\})$ for all $e \in E$.

(4 Points)

Exercise 12.4:
Prove that the Polymatroid Greedy Algorithm, when applied to a vector $c \in \mathbb{R}^E_+$ and a submodular but not necessarily monotone function $f : 2^E \to \mathbb{R}$ with $f(\varnothing) \geq 0$, solves

$$\max\left\{cx \mid \sum_{e \in A} x_e \leq f(A) \text{ for all } A \subseteq E\right\}.$$

(4 Points)

Deadline: Tuesday, January 15, 2013, before the lecture.