Winter term 2012/13 Research Institute for Discrete Mathematics
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Combinatorial Optimization

Exercise Sheet 10

Exercise 10.1:
Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid, $B \subseteq E$, and $J \subset E$ a basis of B. We define $\mathcal{M}/B := (E \setminus B, \{J' \subseteq E \setminus B \mid J' \cup J \in \mathcal{F}\})$. Prove:

1. $\mathcal{M}/B$ is a matroid that does not depend on the choice of J. (2 Points)
2. The rank function of $\mathcal{M}/B$ is given by $r'(A) = r(A \cup B) - r(B)$ for all $A \subseteq E \setminus B$. (2 Points)
3. Let $\varnothing = T_0 \subseteq T_1 \subseteq \ldots \subseteq T_{l+1} = E$. The bases of T_l in $\mathcal{M}$ that intersect T_i in a basis of T_i for each $i \in \{1, \ldots, l\}$ are the bases of T_l in the matroid $\mathcal{N} := \mathcal{N}_0 \oplus \mathcal{N}_1 \oplus \ldots \oplus \mathcal{N}_l$, where $\mathcal{N}_i := (\mathcal{M}/T_i) \setminus \overline{T}_{i+1}$. (4 Points)

Exercise 10.2:
Let $\mathcal{M} = (E, \mathcal{F})$ be a matroid.

1. Let $X \subseteq E$. Prove: Let Y_1 be a base of $\mathcal{M} \setminus X$ and Y_2 be a base of $\mathcal{M}/(E \setminus X)$, then $Y_1 \cup Y_2$ is a base of $\mathcal{M}$. (2 Points)

Now let N and K be nonempty subsets of E. A game $\langle \mathcal{M}; N, K \rangle$ is played as follows: Angelika and Bodo (who plays first) alternatingly tag different elements of N. A tagged element cannot be tagged again later in the game. Angelika wins if she tags a set of elements that span K. Bodo wins if all elements of N are tagged and Angelika did not win.

2. Prove: If N contains two disjoint subsets A_0 and B_0 which span each other and which both span K, then Angelika can win against any strategy Bodo might have. Hint: Assume Bodo picks $a_0 \in A_0$, then there is a $b_0 \in B_0$ such that $(A_0 \setminus \{a_0\}) \cup \{b_0\}$ is a base of $\mathcal{M}_0 := \mathcal{M} \setminus \sigma(A_0 \cup B_0)$... Also, Exercise 10.1 might be helpful. (4 Points)

Note: The other direction is also true, but harder to prove. You may use this fact for the next exercise.

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Now Angelika and Bodo play the even funnier game $\langle G; u, v\rangle$. Here G is a graph and u and v are vertices of G. Angelika and Bodo alternatingly tag edges. Angelika wins if her edges contain a u-v-path and Bodo wins if Angelika didn't win when all edges are tagged. Again Bodo plays first.

3. Prove: Angelika has a winning strategy if and only if there are $V' \subseteq V(G)$, $E_1 \subseteq E(G)$, and $E_2 \subseteq E(G)$ with $\{u, v\} \subseteq V'$ and $E_1 \cap E_2 = \varnothing$ such that (V', E_1) and (V', E_2) are trees. (2 Points)

Deadline: Tuesday, December 18, 2012, before the lecture.