Winter term 2012/13
Juniorprofessor Dr. Stephan Held
Jan Schneider

Research Institute for Discrete Mathematics
University of Bonn

Combinatorial Optimization

Exercise Sheet 8

Exercise 8.1:
Consider the MAXIMUM WEIGHT CUT PROBLEM, i.e. given a graph G and edge weights $c : E(G) \to \mathbb{R}_{>0}$, find a cut $E' \subset E(G)$ with maximum weight $c(E')$. The problem is NP-hard even for $c \equiv 1$.

1. Show that the following algorithm is a 2-factor approximation: Denote $V(G) = \{v_1, \dots, v_n\}$ and set $X := \{v_1\}$. For $i = 3, \dots, n$ add v_i to X if $c(E(\{v_i\}, X)) < c(E(\{v_i\}, \{v_1, \dots, v_{i-1}\} \setminus X))$. (2 Points)

2. Show that this problem can be solved in polynomial time if G is planar.

   Hint: Use Exercise 7.1 and the fact that a connected planar graph is bipartite if and only if its planar dual is Eulerian. (4 Points)

Exercise 8.2:
Let $G = (V, E)$ be a graph that has an ear decomposition $P_0, \dots, P_k$ with $P_0 := (\{r\}, \emptyset)$ and let $T \subseteq V$ with $|T|$ even. For an ear P_i we define the set of inner vertices as $in(P_i) := V(P_i) \setminus \bigcup_{j=0}^{k-1} V(P_j)$. Furthermore, we call an ear P_i pendant if $E(P_i) > 1$ and for every other ear P_j, $i \neq j$, with $E(P_j) > 1$ we have $in(P_i) \cap (V(P_j) \setminus in(P_j)) = \emptyset$ (i.e. no other non-trivial ear is "attached" to an inner vertex of P). We set $\varphi(P_i) := 1$ if $|E(P_i)|$ is even and $\varphi(P_i) := 0$ otherwise.

1. Let P be a pendant ear. Show that there exist $F \subseteq E$ with $|F| \leq \frac{1}{2}|in(P)| + \frac{1}{2}\varphi(P)$ and $S \subseteq V \setminus in(P)$ such that, for every S-join J in $G - in(P)$, $F \cup J$ is a T-join in G. (4 Points)

2. Show that $\tau(G, T) \leq \frac{1}{2}(|V| + \varphi(G) - 1)$, where $\tau(G, T)$ is the minimum cardinality of a T-join in G and $\varphi(G)$ is the minimum possible number of even ears (i.e. ears with $\varphi(P) = 1$) in an ear decomposition of G. (2 Points)

Note: $\varphi(G) = 0$ if and only if G is factor-critical.

Continued on page 2.

Exercise 8.3:
Let $G = (V, E)$ be a graph with edge capacities $u : E \to \mathbb{N} \cup \{\infty\}$ and node labels $b : V \to \mathbb{N}$. For $X, Y \subseteq V$ denote by $q_{(G,u)}(X, Y)$ the number of connected components C in $G - X - Y$ for which $\sum_{v \in V(C)} b(v) + \sum_{e \in E_G(V(C),Y)} u(e)$ is odd. Prove: (G, u) has a perfect b-matching if and only if for any two disjoint subsets $X, Y \subseteq V$

$$q_{(G,u)}(X,Y) \leq \sum_{v \in X} b(v) + \sum_{y \in Y} \left(\sum_{e \in \delta(y)} u(e) - b(y) \right) - \sum_{e \in E_G(X,Y)} u(e).$$

(4 Points)

Deadline: Tuesday, December 4, 2012, before the lecture.