Winter term 2012/13 Research Institute for Discrete Mathematics
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Combinatorial Optimization

Exercise Sheet 6

Exercise 6.1:
We call a directed graph $G = (V, E)$ *special* if the underlying undirected graph is bipartite, loopless, and has no parallel edges, and there exists a permutation $\pi : V \to V$ such that $\pi(v) \neq v$ and $\pi(\pi(v)) = v$ hold for all $v \in V$ and $(v, w) \in E$ implies $(\pi(w), \pi(v)) \in E$. An edge labeling $x : E \to \mathbb{N}_+$ on a special graph is called *balanced* if $x(v, w) = x(\pi(w), \pi(v))$ for every $(v, w) \in E$. If G is a special graph, $u : E \to \mathbb{N}_+$ are edge capacities, and f is a b-flow in (G, u), then a walk P in G is called *valid* if it does not contain both (v, w) and $(\pi(w), \pi(v))$ for some $(v, w) \in E$ such that $u_f(v, w) = u_f(\pi(w), \pi(v)) = 1$.

(i) Prove: Let $G = (V, E)$ be a special graph with balanced edge capacities $u : E \to \mathbb{N}_+$ and f and g different balanced circulations on (G, u). Then there are valid cycles $C_1, \ldots, C_k$ in the residual graph G_f and functions $f_{C_i} : E(C_i) \to \{-1, 0, 1\}$ for $i \in \{1, \ldots, k\}$ such that

$$g(v, w) - f(v, w) = \sum_{i=1}^{k} \left(f_{C_i}(v, w) + f_{C_i}(\pi(w), \pi(v))\right) \quad \text{for all } (v, w) \in E.$$

Hint: Remember the proof of the decomposition theorem for ordinary flows.

Now let (G, u, s, t) be a network with balanced edge capacities u and f a balanced s-t-flow in this network.

(ii) Show that f is a balanced s-t-flow with maximum value if and only if there is no valid s-t-path in G_f.

(iii) Assume that it is possible in polynomial time to find a valid s-t-path in G_f or to decide that none exists. Describe an algorithm that finds a maximum balanced flow in polynomial time.

We want to solve the cardinality matching problem in a simple graph $G = (V, E)$. From this, we construct a directed graph as follows. Let $V' := \{v' \mid v \in V\}$ be a copy of V and $\{s, t\}$ two additional vertices. Furthermore, let $E_G := \{(v, w') \mid \{v, w\} \in E\}$, $E_s := \{(s, v) \mid v \in V\}$, and $E_t := \{(v', t) \mid v \in V\}$, and define the directed graph $D_G := (V \cup V' \cup \{s, t\}, E_G \cup E_s \cup E_t)$. (Continued on next page.)

(iv) Show how the algorithm from (iii) and D_G can be used to find maximum matchings in polynomial time.

(4+2+2+4 Points)

Exercise 6.2:
Let G be a graph and P the fractional perfect matching polytope of G. Prove that the vertices of P are exactly the vectors x with

$$x_e = \begin{cases} \frac{1}{2} & \text{if} e \in E(C_1) \cup \cdots \cup E(C_k) \\ 1 & \text{if } e \in M \\ 0 & \text{otherwise,} \end{cases}$$

where $C_1, \ldots, C_k$ are vertex disjoint odd circuits and M is a perfect matching in $G - (V(C_1) \cup \cdots \cup V(C_k))$. (4 Points)

Exercise 6.3:
Let G be a graph, $T \subseteq V(G)$ with $|T|$ even, and $F \subseteq E(G)$. A subset $C \subseteq E(G)$ is called a T-*cut* if $C = \delta(U)$ for some $U \subseteq V(G)$ with $|U \cap T|$ odd. Prove:

(i) F has nonempty intersection with every T-join if and only if F contains a T-cut.

(ii) F has nonempty intersection with every T-cut if and only if F contains a T-join.

(2+2 Points)

Deadline: Tuesday, November 20, 2012, before the lecture.