Winter term 2012/13 Research Institute for Discrete Mathematics
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Combinatorial Optimization

Exercise Sheet 3

Exercise 3.1:
Prove:

1. A minimal factor-critical graph G has at most $\frac{3}{2}(|V(G)|-1)$ edges. This bound is tight. (Needed for Lemma 1.34) (2 Points)

2. Let G be a graph and M a matching in G. If $X \subseteq V(G)$ is the set of M-exposed vertices, then a shortest M-alternating X-X-walk of positive length can be found in $O(|E(G)|)$. (Lemma 1.39) (2 Points)

Exercise 3.2:
Prove that an undirected graph G is factor-critical if and only if G is connected and $\nu(G) = \nu(G - v)$ for all $v \in V(G)$. (3 Points)

Exercise 3.3:
Let G be a graph and M a matching in G that is not maximum.

1. Show that there are $\nu(G) - |M|$ vertex-disjoint M-augmenting paths in G. *Hint:* Recall the proof of Berge's Theorem.

2. Prove that there exists an M-augmenting path of length at most $\frac{\nu(G)+|M|}{\nu(G)-|M|}$.

3. Let P be a shortest M-augmenting path in G and P' an $(M \triangle E(P))$-augmenting path. Prove $|E(P')| \geq |E(P)| + |E(P \cap P')|$.

Consider the following algorithm: We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \ldots$ be the sequence of augmenting paths chosen.

4. Show that if $|E(P_i)| = |E(P_j)|$ for $i \neq j$, then P_i and P_j are vertex-disjoint.

5. Conclude that the sequence $|E(P_1)|, |E(P_2)|, \ldots$ contains at most $2\sqrt{\nu(G)} + 2$ different numbers. (5 Points)

Exercise 3.4:
Let $G = (V, E)$ a graph and $X \subseteq V$. Let $\beta(G, X)$ be the maximum size of a set $Y \subseteq X$ for which there is a matching in G that covers Y. Prove

$$\beta(G, X) = \min_{U \subseteq V} \{|X| + |U| - q_X(U)\}.$$

Here $q_X(U)$ denotes the number of odd connected components of $(V \setminus U, E)$ whose vertices are all in X.
Hint: Construct a new graph with $2|V|$ vertices and apply Tutte's Theorem. (4 Points)

Deadline: Tuesday, October 30, 2012, before the lecture.